Journal of Engineering Physics and Thermophysics, Vol. 70, No. 2, 1997

A METHOD FOR SOLVING THE HEAT CONDUCTION PROBLEM FOR CONTINUOUSLY DISCRETE ROD SYSTEMS

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UDC 512:536.2

We propose a method for solving problems for thermal processes in complex continuously discrete structures.

One-dimensional boundary-value problems of heat conduction theory with a continuously discrete distribution of thermophysical parameters are of great practical interest, since problems of heat transfer play an important role in the development of many fields of modern technology.

The considerable growth in the power of energy devices with the simultaneous decrease in their dimensions (for example, in space exploration) increases the importance of studying thermal processes in complex systems. Moreover, the reliability of operation and the stability of functioning of various devices are directly determined by thermal processes.

In view of this, so-called continuously discrete boundary-value problems are of particular current interest, due to the need to perform thermal calculations for composite media. Continuosly discrete boundary-value problems represent a mathematical model for investigating the phenomenon of heat transfer in those cases when the physical parameters of the medium under study are not constant for the entire region of its determination, but depend on the coordinates.

We suggest an approach to the solution of such problems when at the boundary between subregions conjugation conditions are introduced that determine certain dependences for the temperature or heat flux. This allows one to reduce the problem to a number of disconnected boundary-value problems.

Let us consider the heat conduction problem for a rod element with a thermally insulated lateral surface of length l that consists of n parts with different thermal conductivity coefficients $\lambda_i(x)$ $(i = \overline{1, n})$ connected by $n-1$ discrete elements. In this case, the thermal conductivity coefficient $\lambda(x)$, the specific heat $c(x)$, and the density $\rho(x)$ are piecewise continuous functions. Moreover, we assume that at a number of points in the rod $x = x_i$ $(i = \overline{1, n-1})$ there are concentrated elements of mass m_i with thermal conductivity coefficients λ_i $(i = \overline{1, n-1})$ and concentrated heat capacities c_i $(i = \overline{1, n-1})$.

The differential heat conduction equation will be written as

$$\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right) - \rho(x)\,c(x)\frac{\partial T}{\partial t} = 0\,, \tag{1}$$

$$x \in G = (0, l) \in R^1\,,\quad G_i = (x_{i-1}, x_i) \subset G\,,$$

$$x_0 = 0\,,\quad x_n = l\,,\quad i = \overline{1, n}\,,\quad t \ge 0\,.$$

This equation must be satisfied over the sections of the rod between the discontinuity points of the functions $\lambda(x)$, $c(x)$, and $\rho(x)$. Suppose the discontinuities of these functions coincide with the points of application of concentrated inclusions m_i. If there are no concentrated inclusions at the points of discontinuity of the functions, it is necessary to set the corresponding value of m_i equal to zero. The assumption of coincidence between the discontinuities of

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 1062-0125/97/7002-0290$18.00

the functions $\lambda(x)$, $c(x)$, and $\rho(x)$ does not restrict the generality of problem solution, since otherwise the magnitude of the discontinuity of a function which is continuous at a given point could be considered to be equal to zero.

In addition to differential equation (1), functions $T(x, t)$ must satisfy at points $x = x_i$ conjugation conditions of the form

$$T_{i+1}(x_i, t) = T_i(x_i, t), \tag{2}$$

$$\frac{\partial T_{i+1}(x_i, t)}{\partial x} - \frac{\partial T_i(x_i, t)}{\partial x} = \frac{m_i c_i}{\lambda_i} \frac{\partial T_i(x_i, t)}{\partial t},$$

$$i = \overline{1, n-1}.$$

The first of the conjugation conditions determines the property of continuity of the solution, and the second, the jump of the derivatives. The second condition can be obtained by integrating heat conduction equation (1) over intervals $[x_i - \varepsilon, x_i + \varepsilon]$ and proceeding to the limit when $\varepsilon \to 0$. Actually,

$$\int_{x_i-\varepsilon}^{x_i+\varepsilon} \frac{\partial T}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right) dx = \int_{x_i-\varepsilon}^{x_i+\varepsilon} \rho(x)\, c(x) \frac{\partial T}{\partial t} dx.$$

Integrating the left-hand side, we obtain

$$\lambda(x)\frac{\partial T}{\partial x}\Big|_{x_i-\varepsilon}^{x_i+\varepsilon} = \int_{x_i-\varepsilon}^{x_i+\varepsilon} \rho(x)\, c(x)\frac{\partial T}{\partial t} dx.$$

On the basis of the theorem of the mean, the right-hand term of this equality, in view of the continuity of the derivative $\partial T(x, t)/\partial t$, can be represented as follows:

$$\int_{x_i-\varepsilon}^{x_i+\varepsilon} \rho(x)\, c(x)\frac{\partial T}{\partial t} dx = \frac{\partial T(x^*, t)}{\partial t} \int_{x_i-\varepsilon}^{x_i+\varepsilon} \rho(x)\, c(x)\, dx,$$

where the integral of the density (of the volumetric mass) $\rho(x)$ determines the density (volumetric mass) of the portion of the rod in the interval $(x_i - \varepsilon, x_i + \varepsilon)$, and x^* is a certain mean point of the same interval. Within the limit for $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} \frac{\partial T(x^*, t)}{\partial t} \int_{x_i-\varepsilon}^{x_i+\varepsilon} \rho(x)\, c(x)\, dx = m_i c_i \frac{\partial T(x^*, t)}{\partial t},$$

because the density (volumetric mass) of the portion is degenerated into a concentrated density (mass) at the point $x = x_i$. In the limiting case we finally obtain the second of the conditions of conjugation (2).

Without disturbing the generality of the proposed method of solution, as boundary conditions we will take the conditions of free heat exchange with an external medium of zero temperature. Then the boundary conditions will be written in the form

$$x = 0: \quad h_1 T(x, t) = \lambda_1 \frac{\partial T(x, t)}{\partial x}, \tag{3}$$

$$x = l: \quad h_n T(x, t) = -\lambda_n \frac{\partial T(x, t)}{\partial x},$$

where h_1 and h_n are the coefficients of external heat conduction and λ_1 and λ_n are the thermal conductivity coefficients of the portions of the rod at the ends $x = 0$ and $x = l$.

To solve boundary-value problem (1)-(3), we apply the method of separation of Fourier variables and seek a solution in the form

$$T(x, t) = X(x)\,\Phi(t)\,. \tag{4}$$

Substituting Eq. (4) into Eq. (1), we obtain two equations

$$\frac{d\Phi}{dt} + \omega^2\Phi = 0\,, \tag{5}$$

$$\frac{d}{dx}\left(\lambda(x)\frac{dX}{dx}\right) + \omega^2\rho(x)\,c(x)\,X = 0\,, \tag{6}$$

where ω^2 is an arbitrary constant.

Since there is no exact solution for ordinary differential equations with variable coefficients (except for particular cases), then to solve Eq. (6) we replace the functions $\lambda(x)$, $\rho(x)$, and $c(x)$ by their mean values at each of the portions. In this case we assume that in the intervals $[x_{i-1}, x_i]$ these functions are rather smooth:

$$\bar{\lambda}_i = \frac{1}{x_i - x_{i-1}}\int\limits_{x_{i-1}}^{x_i}\lambda(x)\,dx\,,\ \ \bar{\rho}_i = \frac{1}{x_i - x_{i-1}}\int\limits_{x_{i-1}}^{x_i}\rho(x)\,dx\,,$$

$$\bar{c}_i = \frac{1}{x_i - x_{i-1}}\int\limits_{x_{i-1}}^{x_i}c(x)\,dx\,,\ \ \bar{\lambda}_i' = \frac{1}{x_i - x_{i-1}}\int\limits_{x_{i-1}}^{x_i}\lambda'(x)\,dx\,.$$

As a result, we arrive at differential equations with constant coefficients:

$$\bar{\lambda}_i\frac{d^2X_i}{dx^2} + \bar{\lambda}_i'\frac{dX_i}{dx} + \omega^2\rho_i c_i X_i = 0\,, \tag{7}$$

$$x \in G = (0, l) \subset R^1\,,\ \ G_i = (x_{i-1}, x_i) \subset G\,,$$

$$x_0 = 0\,,\ \ x_n = l\,,\ \ i = \overline{1, n}\,.$$

With allowance for Eq. (4), conjugation conditions (2) can be written as:

$$X_{i+1}(x_i, \omega) = X_i(x_i, \omega)\,, \tag{8}$$

$$X_{i+1}'(x_i, \omega) - X_i'(x_i, \omega) = -\frac{m_i c_i}{\bar{\lambda}_i}\omega^2 X_i(x_i, \omega)\,,\ \ i = \overline{1, n-1}\,.$$

The boundary conditions take the form

$$x = 0:\ \ h_1X_1(x) - \bar{\lambda}_1X_1'(x) = 0\,,\ \ x = l:\ \ h_nX_n(x) + \bar{\lambda}_nX_n'(x) = 0\,. \tag{9}$$

To solve boundary-value problem (7)-(9), we will use the method of normal fundamental systems of solutions [1]. The characteristic equation for Eq. (7) will have the roots

$$k_{1,2}^{(i)} = \frac{-\dfrac{\bar{\lambda}'}{\bar{\rho}_i \bar{c}_i} \pm \sqrt{\left(\left(\dfrac{\bar{\lambda}'}{\bar{\rho}_i \bar{c}_i}\right)^2 - 4\dfrac{\bar{\lambda}_i}{\bar{\rho}_i \bar{c}_i}\omega^2\right)}}{2\dfrac{\bar{\lambda}_i}{\bar{\rho}_i \bar{c}_i}} . \tag{10}$$

We introduce the following notation:

$$r_i = -\frac{\bar{\lambda}_i'}{2\bar{\lambda}_i}, \quad n_i = \frac{\sqrt{\left(\left(\dfrac{\bar{\lambda}_i'}{\bar{\rho}_i \bar{c}_i}\right)^2 - 4\dfrac{\bar{\lambda}_i}{\bar{\rho}_i \bar{c}_i}\omega^2\right)}}{2\dfrac{\bar{\lambda}_i}{\bar{\rho}_i \bar{c}_i}} .$$

Then the normal fundamental system of solutions will be written as

$$\begin{gathered} \varphi_{1,1}^{(i)}(x, \omega) = \exp(r_i x) \cos n_i (x - x_{i-1}), \quad \varphi_{2,1}^{(i)}(x, \omega) = \frac{\exp(r_i x)}{n_i} \sin n_i (x - x_{i-1}), \\ \varphi_{1,2}^{(i)}(x, \omega) = n_i \exp(r_i x) \sin n_i (x - x_{i-1}), \quad \varphi_{2,2}^{(i)}(x, \omega) = \exp(r_i x) \cos n_i (x - x_{i-1}) . \end{gathered} \tag{11}$$

A general solution of Eq. (7) with the help of system (11) will be obtained in the form

$$\begin{gathered} X_i(x, \omega) = C_1^{(i)} \varphi_{1,1}^{(i)}(x, \omega) + C_2^{(i)} \varphi_{2,1}^{(i)}(x, \omega), \\ X_i'(x, \omega) = C_1^{(i)} \varphi_{1,2}^{(i)}(x, \omega) + C_2^{(i)} \varphi_{2,2}^{(i)}(x, \omega), \end{gathered} \tag{12}$$

$$x \in G = (0, l) \subset R^1, \quad G_i = (x_{i-1}, x_i) \subset G, \quad i = \overline{1, n} .$$

Since system of solutions (11) is a normal fundamental system, we can write

$$C_1^{(i)} = X_i(x_{i-1}, \omega), \quad C_2^{(i)} = X_i'(x_{i-1}, \omega) . \tag{13}$$

With allowance for Eq. (12), from conjugation conditions (8) and relations (13) we have

$$\begin{gathered} C_1^{(i+1)} = C_1^{(i)} \varphi_{1,1}^{(i)}(x_i, \omega) + C_2^{(i)} \varphi_{2,1}^{(i)}(x_i, \omega), \\ C_2^{(i+1)} = C_1^{(i)} \varphi_{1,2}^{(i)}(x_i, \omega) + C_2^{(i)} \varphi_{2,2}^{(i)}(x_i, \omega) - \frac{m_i c_i \omega^2}{\lambda_i} C_1^{(i+1)}, \quad i = \overline{1, n-1} . \end{gathered} \tag{14}$$

From boundary condition (9) at $x = 0$, taking into account Eqs. (11) and (12), we obtain

$$C_1^{(1)} = \frac{\bar{\lambda}_1}{h_1} C_2^{(1)} . \tag{15}$$

From this relation we conclude that all subsequent coefficients $C_1^{(i)}$ and $C_2^{(i)}$ ($i = \overline{1, n}$) can be expressed in terms of the coefficient $C_2^{(1)}$:

$$C_1^{(i)} = C_2^{(1)} u_{1,2}^{(i)}, \quad C_2^{(i)} = C_2^{(1)} u_{2,2}^{(i)}, \quad i = \overline{1, n}, \tag{16}$$

where $u_{1,2}^{(i)}$ and $u_{2,2}^{(i)}$ are as yet unknown coefficients; moreover, it is evident that

$$u_{1,2}^{(1)} = \frac{\bar{\lambda}_1}{h_1}, \quad u_{2,2}^{(1)} = 1 . \tag{17}$$

Substituting Eq. (16) into Eq. (15) and equating the expression at the coefficient $c_2^{(1)}$, we obtain recurrence relations for determing the coefficients:

$$\begin{aligned} u_{1,2}^{(i+1)} &= u_{1,2}^{(i)} \varphi_{1,1}^{(i)} (x_i, \omega) + u_{2,2}^{(i)} \varphi_{2,1}^{(i)} (x_i, \omega) , \\ u_{2,2}^{(i+1)} &= u_{1,2}^{(i)} \varphi_{1,2}^{(i)} (x_i, \omega) + u_{2,2}^{(i)} \varphi_{2,2}^{(i)} (x_i, \omega) - \frac{m_i c_i \omega^2}{\lambda_i} u_{1,2}^{(i+1)} , \quad i = 1, n-1 . \end{aligned} \tag{18}$$

Thus, the general solution of boundary-value problem (7)-(9) will be written in the form

$$\begin{aligned} X_i (x, \omega) &= C_2^{(1)} \left[u_{1,2}^{(i)} \varphi_{1,1}^{(i)} (x, \omega) + u_{2,2}^{(i)} \varphi_{2,1}^{(i)} (x, \omega) \right] = \\ &= C_2^{(1)} \left[u_{1,2}^{(i)} \exp (r_i x) \cos n_i (x - x_{i-1}) + u_{2,2}^{(i)} \exp (r_i x) \sin n_i (x - x_{i-1}) \right] . \end{aligned} \tag{19}$$

Function (19) will satisfy conditions (9) not at all values of ω, but only at the certain so-called eigenvalues of ω_k. To determine these values, from the second boundary condition (9) at $x = x_n = l$ we obtain the following equation:

$$\begin{aligned} & h_n \left[u_{1,2}^{(n)} \exp (r_n l) \cos n_n (l - x_{n-1}) + u_{2,2}^{(n)} n_n \exp (r_n l) \sin n_n (l - x_{n-1}) - \right. \\ & \left. - \bar{\lambda}_n \left[u_{1,2}^{(n)} n_n \exp (r_n l) \sin n_n (l - x_{n-1}) + u_{2,2}^{(n)} \exp (r_n l) \cos n_n (l - x_{n-1}) \right] \right] = 0 . \end{aligned} \tag{20}$$

The eigenvalues of ω_k are found as roots of Eq. (20). Consequently, the solution of boundary-value problem (7)-(9) is determined by formula (19) with allowance for Eq. (20).

From Eq. (5) for each value of ω_k we have

$$\Phi_k (t) = B_k \exp (- \omega_k^2 t) ,$$

where B_k are constants. Then the solution of initial boundary-value problem (1)-(3) for each eigenvalue of ω_k will be determined by

$$\begin{aligned} T_k^{(i)} (x, t, \omega_k) &= X_k^{(i)} (x, \omega_k) \, \Phi_k (t, \omega_k) = \\ &= A_k \exp (- \omega_k^2 t) \left[u_{1,2}^{(i)} \exp (r_i x) \cos n_i (x - x_{i-1}) + \right. \\ &\quad \left. + u_{2,2}^{(i)} n_i \exp (r_i x) \sin n_i (x - x_{i-1}) \right] , \end{aligned} \tag{21}$$

where $A_k = B_k C_k^{(1)}$ are constants.

We will seek a general solution of boundary-value problem (1)-(3) in the form of a superposition of solutions (21)

$$T^{(i)} (x, t) = \sum_{k=1}^{\infty} A_k \exp (- \omega_k^2 t) \, X_k^{(i)} (x, \omega_k) . \tag{22}$$

Assigning the initial condition as

$$T(x, 0) = \kappa(x), \quad \kappa(x) \in C^2(\overline{G}), \tag{23}$$

and allowance for Eq. (22), we can write

$$\sum_{k=1}^{\infty} A_k X_k^{(i)}(x, \omega_k) = \kappa(x). \tag{24}$$

Consequently, the coefficients of the series can be calculated by the Fourier formulas, since equality (24) represents an expansion of the function $\kappa(x)$ into a series in an orthogonal system of functions

$$X_k(x, \omega_k).$$

Preliminarily, we note the following: if we considered a continuous boundary-value problem for Eq. (1) and boundary conditions (3), the orthogonality condition for the functions $X_k(x)$ would have the form

$$\int_0^l \rho(x) X_n(x) X_m(x)\, dx = 0.$$

For continuously discrete boundary-value problem (1)-(3), orthogonality condition takes the following form [2]:

$$\int_0^l \rho(x) X_n(x) X_m(x)\, dx + \sum_{i=1}^{n-1} m_i X_n(x_i) X_m(x_i) = 0.$$

Taking the above into account, we find that

$$A_k = \frac{\int_0^l \kappa(x) \rho(x) X_k(x)\, dx + \sum_{i=1}^{n-1} \kappa(x_i) m_i X_k(x_i)}{\int_0^l \rho(x) X_k^2(x)\, dx + \sum_{i=1}^{n-1} X_k^2(x_i)}. \tag{25}$$

So, considering Eqs. (21), (22), and (25), the solution of the continuously discrete boundary-value problem of heat conduction will be finally written as follows:

$$T(x, t) = \sum_{k=1}^{\infty} \frac{\int_0^l \kappa(x) \rho(x) X_k(x)\, dx + \sum_{i=1}^{n-1} \kappa(x_i) m_i X_k(x_i)}{\int_0^l \rho(x) X_k^2(x)\, dx + \sum_{i=1}^{n-1} m_i X_k^2(x_i)} \times$$

$$\times \left[u_{1,2}^{(i)} \exp(r_i x) \cos n_i (x - x_{i-1}) + u_{2,2}^{(i)} \exp(r_i x)\, n_i \sin n_i (x - x_{i-1}) \right],$$

where the coefficients $u_{1,2}^{(i)}$ and $u_{2,2}^{(i)}$ are determined from recurrence formulas (18).

Thus, the proposed method makes it possible to investigate thermal processes in complex continuously discrete structures. In contrast to other methods, the computation is virtually independent of the number of discrete specific features over the integration interval. For example, the order of characteristic equation (20) does not depend on the number of discrete parameters, but only depends on the boundary conditions of the problem. Recurrence formulas (18) remain the same for all of the basic types of boundary conditions. The method is convenient for programming and computer application.

NOTATION

$\lambda_i(x)$, $c_i(x)$, $\rho_i(x)$, $T_i(x, t)$, thermal conductivity, heat capacity, density, and temperature of the i-th portion of the rod; m_i, mass of the i-th concentrated discrete element; $\overline{\lambda_i}$, $\overline{c_i}$, $\overline{\rho_i}$, average values of the thermal conductivity, heat capacity, and density of the i-th portion of the rod; h_1 and h_2, coefficients of external thermal conductivity at $x = 0$ and $x = l$, respectively; λ_1 and λ_n, thermal conductivity coefficients of the first and the last portions of the rod at points $x = 0$ and $x = l$; λ_i and c_i, thermal conductivity and heat capacity of the i-th concentrated element; $C^2(\overline{G})$, Hilbert space of doubly differentiable functions.

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